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GAUSSIAN AND RELATED RANDOM VARIABLES IN SCHEDULING THEORY

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Abstract. Some scheduling problems are considered and optimal solutions of these problems are described. Stochastic analogs of scheduling problems are discussed and several statements illustrating the usefulness for modeling stochastic parameters of symmetrically truncated Gaussian random variables, as well as their basic properties, are formulated.

Keywords: Combinatorial optimization, a stochastic scheduling problem, a Gaussian (normal) random variable, a symmetrically truncated Gaussian random variable.

Introduction. The term “Scheduling Theory” was introduced by Richard Ernest Bellman (1920–1984) in 1954. We will mainly follow the terminology of monograph of [1].

We consider the following scheduling model: we have one processor (machine), n $(n > 1)$ jobs (numerated by numbers $1,2, \ldots, n$), with release times $r_1, r_2, \ldots, r_n$, processing times $p_1, p_2, \ldots, p_n$ and delivery times $p_1, p_2, \ldots, p_n$. Job j becomes available from its release time r_j and needs continuous processing time p_j on the machine; once completed on the machine, it needs an additional delivery time q_j for its full completion (the delivery of job j is machine-independent and requires no further resource (the job is delivered by an independent agent).

A feasible schedule S is a permutation of the numbers $1,2,\dots,n$. We write $t_j(S)$ for the starting time of job j and $t_j(S)+p_j=c_j(S)$ is the completion time of that job in schedule S. The full completion time of job j in schedule S will be $C_j(S)=c_j(S)+q_j$.

We have $t_j(S)\geq r_j$ and $t_j(S)\geq t_k(S)+p_k$ for any job k included earlier in S.

The objective is to find a feasible schedule S minimizing the maximum job full completion time $C_{max}(S)=\max_{j\leq n}C_j(S)$ and $C_{opt}=\min_{S\in\Pi(J(n))}C_{max}(S)$ (a schedule, assignment with this property is an optimal schedule). The abbreviature $1|r_j,q_j|C_{max}$ was introduced in [3]. It is known that this problem is NP-hard (= non-deterministic polynomial-time hard) [4]. Even in case when the r release times consists of only two elements, this problem is NP-hard (see E.Chinos and N.Vakhania [2, Theorem 1]). Thus, even in this case it is impossible to find the optimal solution in polynomial time. An efficient heuristic method that is commonly used for problem $1|r_j,q_j|C_{max}$ was proposed long time ago by Jackson (1955) for the version of the problem without release times, and then was extended by Schrage (1971) to take job release times into account. The extended Jackson's heuristic (J-heuristic, for short) iteratively, at each scheduling time t (given by job release or completion time), among the jobs released by time t schedules one with the largest delivery time.We will use for the initial J-schedule, i.e. one obtained by the application of Jackson's heuristic to the originally given problem instance, and for an optimal schedule. In case when in the $1|r_j,q_j|C_{max}$ problem the r release times of jobs are identical, then J-heuristic is optimal.

Before we consider the main problem of this work, the stochastic scheduling problem, we will give some auxiliary results from deterministic scheduling problems. The first one is the following:

Proposition 1. If in the problem $1|r_j, q_j|C_{max}$, the released times of jobs are identical $r_1 = r_2 = \cdots = r_n = r$ (the problem $1|q_j|C_{max}$), then the J-schedule is optimal.

Developing the stochastic schedule problems, we simulate the job scheduling process in computer. To get an effective schedule, it is important that the realizations of the stochastic schedules be very close to their mean values. As it is well known, the parameters of the scheduling problem (the job processing times, delivery times, etc.) are random variables. Since they are the sums of many independent random variables, by the central limit theorem, their probability distributions are close to the distributions of Gaussian random variables. Indeed, let the random variable ξ_1 be with mean value p and variance σ^2. Let us consider independent copies $\xi_1, \xi_2, \dots, \xi_n, \dots$ and form the following random variable

$$(\xi_1 + \xi_2 + \cdots + \xi_n - np)/(\sqrt{n}\sigma), \ 1,2, \dots$$

By the central limit theorem this sequence converges in distribution to the standard Gaussian random variable with mean 0 and variance 1.

If for the simulation of the stochastic scheduling process we will use Gaussian random variables, it is possible to get a negative value for processing time or delivery time. This may happen because a Gaussian random variable can take values that are less than a fixed arbitrary small negative number with a positive probability. For this reason, in stochastic scheduling problem we consider the symmetrically truncated Gaussian random variable instead of Gaussian random variables. Let us illustrate this statement by the following

Example. Let the mean value of the stochastic processing time of the job be $p = 5$ hour. If we simulate this value by the Gaussian random variable $p = 5 + \gamma(0, \sigma)$, where $\gamma(0, \sigma)$ is Gaussian random variable with mean 0 and variance σ^2, we can get negative values as $P\{\gamma(0, \sigma) < -5\} > 0$. What happens in case of our notion? Let us take the truncation level M, such that $P\{|\gamma(0, \sigma, M)| \leq M\} = 1$, where $\gamma(0, \sigma, M)$ is symmetrically truncated Gaussian random variable (see definition and properties bellow). If we take the number 3 as M, we get positive values for processing time.

Symmetrically truncated Gaussian random variable. Let $\gamma(\sigma) \equiv \gamma(o,\sigma)$ be Gaussian random variable with mean 0 and variance σ^2. The distribution function of this random variable is

$$F_{\gamma(\sigma)}(t) \equiv P(\gamma(\sigma) \le t) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{t} e^{\frac{-x^2}{2\sigma^2}}\,dx.$$

For any positive numbers M and σ, denote by $\Gamma(\sigma, M)$ the random variable with following distribution function

$$F_{\sigma,M} \equiv F_{\Gamma(\sigma,M)}(t) \equiv P(\Gamma(\sigma,M) \le t) = \frac{1}{\sqrt{2\pi}\sigma}\rho\int_{-M}^{t} e^{\frac{-x^2}{2\sigma^2}}\,dx, -M \le t \le M,$$

where

$$\rho = \frac{1}{F_{\gamma(\sigma)}(M) - F_{\gamma(\sigma)}(-M)} = \frac{\sqrt{2\pi}\sigma}{\int_{-M}^{M} e^{\frac{-x^2}{2\sigma^2}}dx}.$$

The distribution density of the random variable $\Gamma(\sigma, M)$ is

$$f_{\sigma,M}(x) \equiv f_{\Gamma(\sigma,M)}(x) = \frac{1}{\sqrt{2\pi}\sigma}\rho e^{\frac{-x^2}{2\sigma^2}} =$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-x^2}{2\sigma^2}}\sqrt{2\pi}\sigma\left(\int_{-M}^{M} e^{\frac{-x^2}{2\sigma^2}}\,dx\right)^{-1} = \frac{e^{\frac{-x^2}{2\sigma^2}}}{\int_{-M}^{M} e^{\frac{-y}{2\sigma^2}}dy}, -M \le x \le M.$$

Such random variable we call a symmetrically truncated Gaussian random variable with bound (or level) M and parameter σ.

Below we provide some properties of this random variable.

Proposition 2. For any fixed M, the sequence of densities of symmetrically truncated Gaussian random variables with parameters $\sigma_n$tends to the density of uniformly distributed random variable when σ_n tends to infinity

$$\lim_{\sigma_n \to \infty} f_{\sigma_n,M}(x) \to U_{[-M,M]}(x),$$

uniformly in x on the segment $[-M, M]$, where $U_{[-M,M]}(x)$ is the density of uniformly distributed random variable.

Proposition 3. Let $\gamma_1(\sigma, M)$ and $\gamma_2(\sigma, M)$ be independent symmetrically truncated Gaussian random variables, ξ_1 and ξ_2 be independent Gaussian random variables with mean 0 and variance σ^2; denote $S = \xi_1 + \xi_2$. If $M \ge \sigma$, then

$$E\big(\gamma_1(\sigma,M)+\gamma_2(\sigma,M)\big)^2 \le\ E(S\big(\sqrt{2}\sigma,2M\big)^2.$$

Analogously, the following general statement holds:

Proposition 4. Let $\gamma_1(\sigma,M),\gamma_2(\sigma,M),\dots,\gamma_n(\sigma,M)$ be independent symmetrically truncated Gaussian random variables, $\xi_1,\xi_2,\dots,\xi_n$ be independent Gaussian random variables with mean 0 and variance σ^2; denote $S_n\equiv\xi_1+\xi_2+\cdots,\xi_n$. If $\frac{M}{\sigma}>\sqrt{\frac{\ln n}{n-1}}$ then

$$E\ (\gamma_1(\sigma,M)+\gamma_2(\sigma,M)+\cdots+\gamma_n(\sigma,M))^2\le E\left(S_n\big(\sqrt{n}\sigma,nM\big)\right)^2.$$

Proposition 5. Let $0<\sigma<\theta$. If $-M<x<-x_0$ or $M>x>x_0$, then $f_{\sigma,M}<f_{\theta,M}$ and if $-M<-x_0<x<x_0<M$, then $f_{\sigma,M}>f_{\theta,M}$. If $-x_0<-M<x<M<x_0$, then $f_{\sigma,M}>f_{\theta,M}$.

Let in the Borel σ-algebra of R^1 we have the probability distribution $P_{\gamma(\sigma)}$ of the Gaussian random variable $\gamma(\sigma)$. For any fixed $M>0$, let us consider the Borel σ-algebra B_M of the segment $[-M,M]$. The conditional expectation of the indicator of the measurable subset A of R^1 with respect to the σ-algebra B_M equals

$$E(I_A|B_M)=P_{\gamma(\sigma)}(A|B_M)=\frac{P_{\gamma(\sigma)}(A\cap[-M,M])}{P_{\gamma(\sigma)}([-M,M])}=P_{\gamma(\sigma),M}(A).$$

For any arbitrary probability space (Ω,B,P) it is meaningful to define the conditional expectation $E(\xi|B_E)$ for any measurable set E with $P(E)>0$ where B_E is a σ-algebra of E induced by B and for any random variable ξ defined on (Ω,B,P).

$E(\xi|B_E)$ is B_E-measurable random variable, such that $\int_A E(\xi|B_E)\,dP=\frac{\int_A\xi dP}{P(E)}$.

Let us consider the Gaussian random variable $\gamma(\sigma)$, any positive number M and the measurable set $E=[|\gamma(\sigma)|\le M]$. Denote by B_E theσ-algebra induced from B, to the set E.

Proposition 6. Let $\gamma(\sigma)$ be Gaussian random variable with mean 0 and variance σ^2, M be any positive number. The corresponding symmetrically truncated Gaussian random variable $\gamma(\sigma,M)$ can be obtained as a conditional expectation of $\gamma(\sigma)$ by the Borel σ-algebraB_E on the set $E=[|\gamma(\sigma)|\le M]$.

Stochastic scheduling process. Let us consider the stochastic scheduling process when we have jobs $j_1, j_2, \dots, j_n$, with released times $r_1 = r_2 = \cdots = r_n = 0$, processing times are independent random variables with mean p_i added symmetrically truncated Gaussian random variable $\gamma_i(\sigma, M), i = 1,2, \dots, n$. Delivery times also are independent to each other and independent to processing times random variables with mean q_i added symmetrically truncated Gaussian random variable $\gamma_i(\theta, L), i = 1,2, \dots, n$.

A feasible schedule S is a sequence of ordering jobs $j_1(S), j_2(S), \dots, j_n(S)$. The full completion time of job j in schedule S, $C_j(S) = p_1(S) + \gamma_1(S)(\sigma, M) + \cdots + p_j(S) + \gamma_j(\sigma, M) + q_j(S) + \gamma_j(\theta, L)$. The objective is to find a feasible schedule S minimizing the maximum job full completion time $C_{max}(S) = \max_{j \le n} EC_j(S)$ and $C_{opt} = \min_{S \in \Pi(J(n))} C_{max}(S) = \max_{j \le n} EC_j(S)$.

Theorem 1. In the stochastic job scheduling problem with released times $r_1 = r_2 = \cdots = r_n = 0$, stochastic processing times with mean value p_i plus independent symmetrically truncated Gaussian random variables with parameters σ and M and stochastic delivery times with mean value q_i plus independent, symmetrically truncated Gaussian random variables with parameters θ and L, the optimal schedule in an average sense is the J-schedule ($q_1 \ge q_2 \ge \cdots \ge q_n$).

Thus, the optimal schedule in an average sense is the J-schedule. It is interesting what difference is with optimal schedule in an average sense and real optimal schedule.

At first, let us consider the case, when the delivery times are deterministic (case a).

Theorem 2. In the stochastic job scheduling problem with released times $r_1 = r_2 = \cdots = r_n = 0$, stochastic processing times with mean value p_i plus independent, symmetrically truncated Gaussian random variables with parameters σ and M and with deterministic delivery times q_i, the J-schedule is optimal and quantity of difference with average optimal full processing time C_{opt} and real optimal full processing time C_{opt}^r is

$$\left|C_{opt}^r - C_{opt}\right| \le |\gamma_1(\sigma, M) + \gamma_2(\sigma, M) + \cdots + \gamma_n(\sigma, M)|.$$

Let us consider now the case when the delivery times are mean value plus symmetrically truncated Gaussian random variables and processing times are deterministic (case b).

Theorem 3. In the stochastic job scheduling problem with released times $r_1 = r_2 = \cdots = r_n = 0$, deterministic processing times and stochastic delivery times with mean value q_i plus independent, symmetrically truncated Gaussian random variables with parameters θ and L the J-schedule is not, in general, optimal and quantity of difference with average optimal full processing time C_{opt} and real optimal full processing time C_{opt}^r of the optimal scheduling C_{opt}^r is

$$\left|C_{opt}^r - C_\sigma\right| \leq \max_{i \leq n}|\gamma_{j_i}(\theta, L)| \leq L.$$

Let us consider now the case when job processing times and delivery times are random variables (case c).

Theorem 4. In the stochastic job scheduling problem with released times $r_1 = r_2 = \cdots = r_n = 0$, stochastic processing times with mean value p_i plus symmetrically truncated independent Gaussian random variables with parameters σ and M and stochastic delivery times with mean value q_i plus independent, symmetrically truncated Gaussian random variables with parameters θ and L the J-schedule is not, in general, optimal and quantity of difference with optimal full processing time C_{opt} and real optimal full processing time C_{opt}^r of optimal scheduling is

$$\left|C_{opt}^r - C_\sigma\right| \leq \max_{i \leq n}|\gamma_{j_i}(\theta, L)| \leq L.$$

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